

# Binary Curves of Fixed Genus and Gonality with Many Points

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This is ongoing joint work with Xander Faber.



## manypoints.org

• For given (small) finite field and (small) genus, what is the maximum number of points a smooth, projective curve can have?



## manypoints.org

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- There is a fairly extensive database at manypoints.org.
- For  $\mathbb{F}_2$  it looks like:

	g	$N_2(g)$
	0	3
	1	5
	2	6
	3	7
	2 3 4 5	8 9
	5	9





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- The gonality γ of a curve X over a field k is the minimum degree of a k-morphism X → P<sup>1</sup>.
- Gonality 1 curves are isomorphic to P<sup>1</sup>, so coincide with genus 0 curves.
- Gonality 2 curves are **hyperelliptic**, and include elliptic curves (genus 1 and up).
- Gonality 3 curves are known as trigonal curves.



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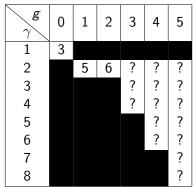
or

• It's fun and interesting.



#### Let's start a table for binary curves

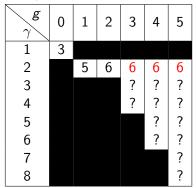
• Proposition: If g = 0, then  $\gamma = 1$ . If g = 1, then  $\gamma = 2$ . If  $g \ge 2$ , then  $\gamma \le 2g - 2$ .





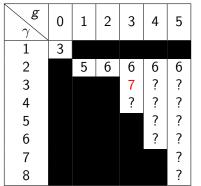
## Hyperelliptic curves

- Proposition: The number of points on a binary curve of gonality γ is ≤ 3γ.
- Proposition: For each genus g ≥ 2, there exists a hyperelliptic curve over 𝔽<sub>2</sub> with 6 rational points.
- Proof: Look at  $y^2 + [1 + x^g(x+1)] \ y = [x(x+1)]^{g-\delta} = 0$ , where  $\delta = g \pmod{2}$ .





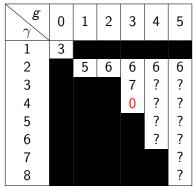
• Proposition: For a curve with rational points,  $\gamma \leq g$ .





## Genus 3, Gonality 4

- A genus-3 curve with rational points must have gonality  $\leq$  3.
- $(x^2 + xz)^2 + (x^2 + xz)(y^2 + yz) + (y^2 + yz)^2 + z^4 = 0$  has gonality 4.
- $\bullet$  Demonstrate lack of degree-3 morphism by looking at  $\mathbb{F}_4$  points.





#### Fun facts about genus 4 non-hyperelliptic curves

- Non-hyperelliptic curves of genus 4 can be embedded as the intersection of a quadric surface and a cubic surface.
- If the quadric surface is xy + zw = 0 or  $xy + z^2 = 0$ , the curve is trigonal.
- If the quadric surface is  $xy + z^2 + wz + w^2 = 0$ , the curve is not.



• Consider the curve:

$$xy + zw = 0$$
  
$$xy^{2} + y^{3} + x^{2}z + y^{2}z + xz^{2} + x^{2}w + y^{2}w + xw^{2} = 0.$$

• It has 8 points and is trigonal.



- The surface  $xy + z^2 + wz + w^2 = 0$  has only 5 rational points.
- Consider the curve:

$$xy + z^{2} + zw + w^{2} = 0$$
  
$$xy^{2} + x^{2}z + y^{2}z + yz^{2} + x^{2}w + z^{2}w = 0.$$

• It has 5 points.



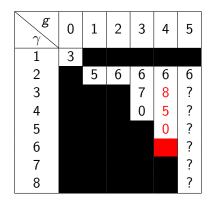
- If a genus 4 curve has gonality 5 or 6, it must be pointless.
- Consider the curve:

$$xy + z^{2} + zw + w^{2} = 0$$
  
$$x^{3} + y^{3} + z^{3} + y^{2}w + xzw = 0.$$

- Not gonality 4; look at  $\mathbb{F}_{16}$  points.
- Write down degree-5 morphism.
- Do this for all pointless curves to rule out gonality 6. (Computations!)



#### Updated Table





- Trigonal curves of genus 5 are birationally equivalent to plane quintics with a multiplicity-2 singularity.
- This gives an upper bound of 2<sup>2</sup> + 2 + 1 + 1 (based on the size of ℙ<sup>2</sup>).
- We achieve this bound with

$$xyz^{3} + x^{3}z^{2} + y^{3}z^{2} + x^{4}z + xy^{3}z + y^{4}z + x^{4}y + x^{2}y^{3} = 0.$$

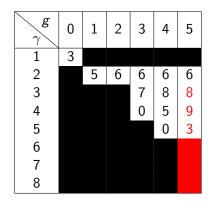
• The genus-5 curves on manypoints.org with 9 points have degree-4 morphisms and thus gonality 4.



- Non-hyperelliptic, non-trigonal genus-5 curves are intersections of three quadric surfaces.
- Can exhaust over pointless genus-5 curves to rule out gonality above 5.
- Can exhaust over all genus-5 curves by looking at possible divisors to show that the maximum number of points on a gonality-5 curve is 3.



#### Current Table





- Refine gonality computation code and algorithm.
- Ternary curves.
- gonality.org

