Goldbach Variations

Jon Grantham
Hester Graves

Institute for Defense Analyses
Center for Computing Sciences
Bowie, Maryland

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Goldbach's Other Other Conjecture

- The conjecture that prompted this research as follows:
- Let A be the set of numbers a for which $a^2 + 1$ is prime. Then every $a \in A$ $(a > 1)$ can be written in the form $a = b + c$, for $b, c \in A$.

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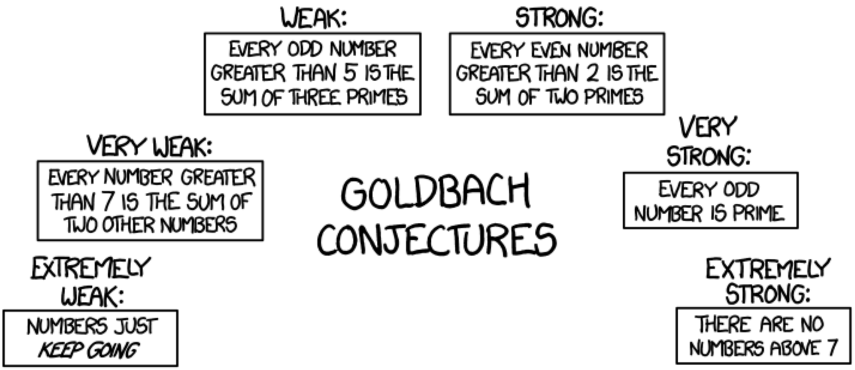
- Let A be the set of numbers a for which $a^2 + 1$ is prime. Then every $a \in A$ $(a > 1)$ can be written in the form $a = b + c$, for $b, c \in A$.
- This comes from a October 1, 1742 letter from Goldbach to Euler.

obxkcd


WEAK:
EVERY ODD NUMBER GREATER THAN 5 IS THE SUM OF THREE PRIMES
STRONG:
EVERY EVEN NUMBER GREATER THAN 2 IS THE SUM OF TWO PRIMES
VERY WEAK:
EVERY NUMBER GREATER THAN 7 IS THE SUM OF TWO OTHER NUMBERS
GOLDBACH CONJECTURES
VERY STRONG:
EVERY ODD NUMBER IS PRIME
EXTREMELY WEAK:
NUMBERS JUST KEEP GOING
EXTREMELY STRONG:
THERE ARE NO NUMBERS ABOVE 7

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Computations

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- I noted that a table up to 10^{25} had been computed by Wolf and Gerbicz (2010).
- I said, "I can beat that."

Results of Computation

- Let $\pi_q(x)$ be the number of primes of the form $a^2 + 1$ up to x.
- $\pi_q(10^{26}) = 237542444180$.
- $\pi_q(10^{27}) = 722354138859$.
- $\pi_q(10^{28}) = 2199894223892$.
- $\pi_q(6.25 \times 10^{28}) = 5342656862803$.

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- I.e. for a up to 10^{14}.
- How do you confirm it, given this 30+ terabyte list?
- Let a_n be the nth integer such that $a_n^2 + 1$ is prime.
- Is $a_n - a_{n-1} = a_i$ for some i? How about $a_n - a_{n-2}$?
- How far back do you have to go?

Large values of $j(a_n)$

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- $j(74) = 3$.
- $j(384) = 6$.
- $j(860) = 7$.
- $j(1614) = 10$.
- $j(7304) = 12$.
- $j(14774) = 14$.
- $j(37884) = 17$.
- $j(103876) = 21$.
- $j(191674) = 23$.
- $j(651524) = 24$.

Even larger values of $j(a_n)$

- $j(681474) = 26.$
- $j(1174484) = 38.$
- $j(10564474) = 44.$
- $j(19164094) = 48.$
- $j(30294044) = 52.$
- $j(279973066) = 56.$
- $j(709924604) = 58.$
- $j(2043908624) = 64.$
- $j(2381625424) = 65.$
- $j(4862417304) = 69.$
- $j(8476270536) = 70.$
- $j(10835743444) = 71.$
- $j(58917940844) = 83.$
- $j(88874251714) = 90.$
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- Schinzel's Hypothesis H (1958):
- Take a set of polynomials $f_i(x)$ such that there is no p for which $\prod f_i(a) \equiv 0$ for all $a \in \mathbb{F}_p$.
- The polynomials are simultaneously prime for infinitely many values of x.

How often is $j(a_n) > 1$?

- Let $f_1(y) = (65y + 9)^2 + 1$ and $f_2(y) = (65y + 1)^2 + 1$.
- Both will be prime simultaneously infinitely often, assuming Hypothesis H.
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- Yes!
- $(65y+3)^2 + 1 \equiv 0 \bmod 5$.
- $(65y+5)^2 + 1 \equiv 0 \bmod 13$.
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- Any such $f_1(x) - f_2(x) = 8 \notin A$.
- So $j(a_n) > 1$ infinitely often.

Growth of $j(a_n)$

- Assuming Hypothesis H, a more complicated version of this argument gives $\limsup_{n\to\infty} j(a_n) = \infty$.
- It is easy to form the polynomials, but mildly tricky to ensure that the polynomials aren't identically zero for some p.

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- It is easy to form the polynomials, but mildly tricky to ensure that the polynomials aren't identically zero for some p.
- A less complicated version of this argument gives $\liminf_{n\to\infty} j(a_n) = 1$.

A Generalization

- It is fun to verify the conjecture of Goldbach.
- However, Michael Filaseta asked about a comparison of $A + A$ to $2\mathbb{Z}$ instead of just to A.
- In fact, we quickly verify this up to 10^{11}.
- Conjecture: Let A be the set of numbers a for which $a^2 + 1$ is prime. Then every $a \in 2\mathbb{Z}$ $(a > 0)$ can be written in the form $a = b + c$, for $b, c \in A$.

A More General Generalization

- What if we think of $x^2 + 1$ as the 4th cyclotomic polynomial?

- Conjecture: Let $\phi_k(x)$ be the kth cyclotomic polynomial.
- Let A_k be the set of positive integers such that $\phi_k(x)$ is prime.
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- Then any sufficiently large even integer can be written as the sum of two elements in A_k.
- For $k = 1$ and $k = 2$ this is actually the more famous Goldbach conjecture!
- For $k = 4$, this is the less–famous conjecture we had been studying.

More computations

- Fortunately, $\phi_3(x-1) = \phi_6(x)$, so we can do two computations for the price of one.
- As far as we can tell, the only computations of primes of this type are due to Luigi Poletti in 1929.
- Google him later. Seriously.
- We are in the process of extending these computations and doing higher-degree computations.