Repeatedly Appending Digits and Only Finding Composites

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Illustrating the Question

- 11 is prime.

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- 21 is composite.

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- 51 is composite.

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- 511 is composite.

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- 41 is prime.
- 5111 is composite.

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- 911 is prime.
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- In general, for every b, k and d, is there a positive integer n such that $s_n^{d,b}(k)$ is prime?
- No.

Base 10, Digit 1

- A 2011 article in the *American Math. Monthly* by Lenny Jones gives the example of 37, where for all $n > 0$, $s_n^{1,10}(37)$ is composite.
- He showed that 37 is minimal by exhibiting values of n such that $s_n^{1,10}(k)$ is prime for all $1 \leq k \leq 36$.

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- What happens for 37? When $b = 10$, $d = 1$,
- If $n \equiv 0 \pmod 3$, $s_n^{1,10}(k) \equiv k \pmod{37}$.
- If $n \equiv 2 \pmod 3$, $s_n^{1,10}(k) \equiv k + 2 \pmod 3$.
- If $n \equiv 1 \pmod 6$, $s_n^{1,10}(k) \equiv 3k + 1 \pmod 7$.
- If $n \equiv 4 \pmod 6$, $s_n^{1,10}(k) \equiv 3k + 6 \pmod{13}$.

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- If $n \equiv 1 \pmod{6}$, $s_n^{1,10}(k) \equiv 3k + 1 \pmod{7}$.
- If $n \equiv 4 \pmod{6}$, $s_n^{1,10}(k) \equiv 3k + 6 \pmod{13}$.
- In other words, **covering congruences** ensure that each $s_n^{1,10}(37)$ is divisible by one of these four primes.

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- For every $k < 4070$, we exhibit a prime value of $s_n^{3,10}(k)$, except when $k = 817$.
- The value $s_n^{3,10}(817)$ is composite for $1 \leq n \leq 554,789$.
- But factorizations show no apparent obstruction to primality, so we conjecture that 4070 is minimal for digit 3.

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- We find the same for 891.
- We find primes for $k < 891$, except when $k = 480$ or $k = 851$.
- The values $s_{11330}^{7,10}(480)$ and $s_{28895}^{7,10}(851)$ have each passed 200 strong pseudoprime tests.
- (It took 45 hours to prove the primality of $s_{2904}^{7,10}(9)$.)

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- Riesel Project primarily concerned with $b = 2$.
- It is known $s_n^{1,2}(509202)$ is always composite.
- There are 55 values of $k < 509202$ with no known primes.
- See www.noprimeleftbehind.net, among others.

Base m^2 (m odd), Digit 1

- Covering congruences are not the only tool.
- When n is even we have:
- $s_n^{1,m^2}(1) = \frac{{m^2}^{n+1}-1}{m^2-1} = \left(\frac{m^{n+1}-1}{m-1}\right)\left(\frac{m^{n+1}+1}{m+1}\right)$.
- When n is odd, we have divisibility by m^2+1 (and hence 2).
- So 1 is always minimal for square bases.

Other Bases, Other Web Sites

- For other bases up to 10, see John Rickert's site:
- http://www.rose-hulman.edu/~rickert/Compositeseq

Pandigital

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- Is there a k such that $s_n^{d,10}(k)$ is composite for all n and for each of $d = 1$, 3, 7, and 9?
- Yes. $k = 4942768284976776320$.
- A more involved covering congruence argument is involved.
- In fact, we show there are infinitely many.

Open Problems

- Find a prime value of $s_n^{3,10}(817)$.
- Find a prime value of $s_n^{9,10}(4420)$.
- Find a prime value of $s_n^{9,10}(7018)$.
- Prove primality of $s_{11330}^{7,10}(480)$ and $s_{28895}^{7,10}(851)$.
- Similar problems for other bases (see Rickert's web site).
- Find a smaller number that works for all digits.