Yet Another Conjecture of Goldbach: Preliminary Results

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Goldbach's Other Other Conjecture

- The conjecture I'm talking about is as follows:
- Let A be the set of numbers a for which $a^2 + 1$ is prime. Then every $a \in A$ $(a > 1)$ can be written in the form $a = b + c$, for $b, c \in A$.

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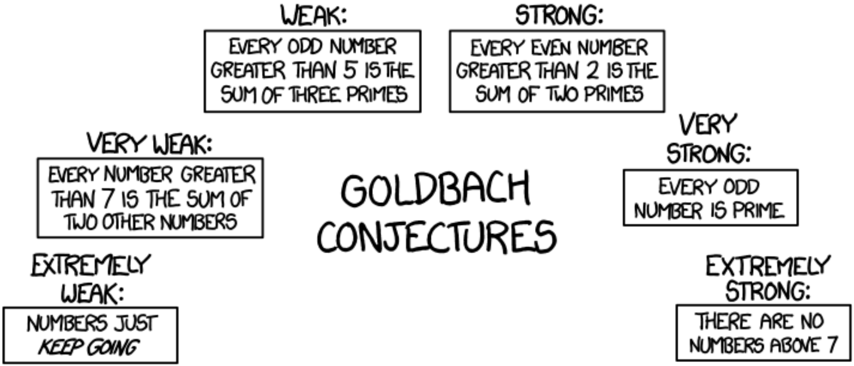
- Let A be the set of numbers a for which $a^2 + 1$ is prime. Then every $a \in A$ $(a > 1)$ can be written in the form $a = b + c$, for $b, c \in A$.
- This comes from a October 1, 1742 letter from Goldbach to Euler.

obxkcd



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Computations

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- I noted that a table up to 10^{25} had been computed by Wolf and Gerbicz (2010).
- I said, "I can beat that."

Results of Computation

- Let $\pi_q(x)$ be the number of primes of the form $a^2 + 1$ up to x.
- $\pi_q(10^{26}) = 237542444180$.
- $\pi_q(10^{27}) = 722354138859$.
- $\pi_q(10^{28}) = 2199894223892$.
- $\pi_q(6.25 \times 10^{28}) = 5342656862803$.
- I talked about this at the first MASON.

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- We have confirmed Goldbach's conjecture up to 10^{28}.
- I.e. for a up to 10^{14}.
- How do you confirm it, given this 30+ terabyte list?
- Let a_n be the nth integer such that $a_n^2 + 1$ is prime.
- Is $a_n - a_{n-1} = a_i$ for some i? How about $a_n - a_{n-2}$?
- How far back do you have to go?

Large values of $j(a_n)$

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- Let's look at champion values.
- $j(74) = 3$.
- $j(384) = 6$.
- $j(860) = 7$.
- $j(1614) = 10$.
- $j(7304) = 12$.
- $j(14774) = 14$.
- $j(37884) = 17$.
- $j(103876) = 21$.
- $j(191674) = 23$.
- $j(651524) = 24$.

Even larger values of $j(a_n)$

- $j(681474) = 26$.
- $j(1174484) = 38$.
- $j(10564474) = 44$.
- $j(19164094) = 48$.
- $j(30294044) = 52$.
- $j(279973066) = 56$.
- $j(709924604) = 58$.
- $j(2043908624) = 64$.
- $j(2381625424) = 65$.
- $j(4862417304) = 69$.
- $j(8476270536) = 70$.
- $j(10835743444) = 71$.
- $j(58917940844) = 83$.
- $j(88874251714) = 90$.
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- Schinzel's Hypothesis H (1958):
- Take a set of polynomials $f_i(x)$ such that there is no p for which $\prod f_i(a) \equiv 0$ for all $a \in \mathbb{F}_p$.
- The polynomials are simultaneously prime for infinitely many values of x.

How often is $j(a_n) > 1$?

- Let $f_1(y) = (65y+9)^2 + 1$ and $f_2(y) = (65y+1)^2 + 1$.
- Both will be prime simultaneously infinitely often, assuming Hypothesis H.
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- Yes!
- $(65y+3)^2+1 \equiv 0 \bmod 5$.
- $(65y+5)^2+1 \equiv 0 \bmod 13$.
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- Any such $f_1(x) - f_2(x) = 8 \notin A$.
- So $j(a_n) > 1$ infinitely often.

Growth of $j(a_n)$

- Assuming Hypothesis H, a more complicated version of this argument gives $\limsup_{n\to\infty} j(a_n) = \infty$.
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- A less complicated version of this argument gives $\liminf_{n\to\infty} j(a_n) = 1$.

Future Work

- Apply Bateman–Horn conjecture to get explicit bounds on $j(a_n)$.
- Conjecture growth of average values of $j(a_n)$.
- Conjecture growth of champion values of $j(a_n)$.