Repeatedly Appending Any Digit to Generate Composite Numbers

Jon Grantham, Witold Jarnicki, John Rickert, and Stan Wagon

Abstract. We investigate the problem of finding integers k such that appending any number of copies of the base-ten digit d to k yields a composite number. In particular, we prove that there exist infinitely many integers coprime to all digits such that repeatedly appending *any* digit yields a composite number.

1. INTRODUCTION. Recently, L. Jones [**5**] asked about integers that yield only composites when a sequence of the same base-ten digit is appended to the right. He showed that 37 is the smallest number with this property when appending the digit $d = 1$. For each digit $d \in \{3, 7, 9\}$, he also found numbers coprime to d that yield only composites upon appending ds.

In this paper, we find a single integer that works for all digits simultaneously. More precisely, we prove the following.

Theorem. *There are infinitely many positive integers k with* $\gcd(k, 2 \cdot 3 \cdot 5 \cdot 7) = 1$*, such that for any base-ten digit d, appending any number of ds to k yields a composite number.*

Further, we investigate the question of the smallest numbers that remain composite upon appending strings of a digit for each particular digit. Jones found, for digits 3, 7, 9, respectively, the examples 4070, 606474, and 1879711. It appears that 4070 is the smallest for $d = 3$; for digit 7 we found 891, which is almost certainly minimal; and for digit 9, the likely answer 10175 was discovered by [**14**]. In the next section, we explain the obstructions to proving that these three answers are the smallest.

2. SEEDS. Given a digit d, let's use the term *seed* for a number coprime to d such that appending any number of ds on the right yields a composite. The smallest positive integer with this property will be referred to as a *minimal seed*. Only the cases $d \in \{1, 3, 7, 9\}$ are nontrivial. Jones proved that 37 is the minimal seed for $d = 1$, and he also found the seed 4070 for digit 3. For every $k < 4070$, except 817, we have found a value of n such that appending n 3s yields a prime or, in three cases, a probable prime. For 817, appending up to 554789 3s yielded only composites. But factorizations show no apparent obstruction to primality, so we conjecture that 4070 is the minimal seed for digit 3.

A key concept in this area is the notion of a covering set, introduced by P. Erdős [**3**]. Such a set corresponds to a finite list of primes such that every member of a given sequence is divisible by one of the primes. Here the sequences are the numbers, which we call s_n, obtained by appending n copies of a digit d to an initial value k; typically, the numbers are proved composite by finding a covering set. For example, when n 7s are appended to 891, the resulting number is divisible by 11, 37, 11, 3, 11, or 13 according to the mod-6 residue of n (starting at 0).

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To see this, observe that s_n is given by the formula

$$s_n = k \cdot 10^n + \frac{d(10^n - 1)}{9}.$$

Because $10^6 \equiv 1$ modulo each of the four primes, easy modular arithmetic shows that $s_{6m+i} \equiv 0 \pmod{p}$ for the cases $p = 11$, 13, and 37, where i, depending on p, is 0, 2, 4, 5, or 1. The same is true for $i = 3$, the case where $p = 3$, because $10^{6m+3} - 1$ is divisible by 27, thus eliminating the denominator of 9 in these cases. This proves that 891 is a seed for digit 7.

When a sequence of primes $(p_0, p_1, \ldots, p_{r-1})$ divides the corresponding sequence of terms s_n for a digit d and seed k, we say that the primes form a *prime cover* for (k, d). For example, $(11, 37, 11, 3, 11, 13)$ is a prime cover for $(891, 7)$.

We have shown that 891 is a minimal seed for digit 7, under the assumption that appending 11330 7s to 480, and 28895 7s to 851 yields primes. Each of these two large numbers has passed 200 strong pseudoprime tests. For all other potential seeds below 891, we have found primes that can be certified using elliptic curve methods with *Mathematica* or *Primo* [**9**]. We used *Primo* on the largest cases; the largest was 9777...7 with 2904 7s, which took 45 hours.

The digit-9 case asks for an integer k such that $(k + 1)10^n - 1$ is always composite; it is thus a variation on the classic Riesel problem [**7, 11, 12, 13**], which addresses the same question in base 2. For that classic case, it is known that 509202 is a seed, meaning that $509203 \cdot 2^n - 1$ is composite for $n \geq 0$. Participants in the Riesel project have also investigated the decimal case, and showed [**14**] that the expected minimal seed for digit 9 is 10175. To see that this is a seed, we again consider the number of appended digits modulo 6 and find a prime cover: in this case $(11, 7, 11, 37, 11, 13)$. Of the numbers smaller than 10175, only 4420 has not been eliminated as a seed. The Riesel project [**12, 13**] has checked it through the addition of 940000 9s without finding a prime. In this case, primality proving for a probable prime is easy using the Lucas $n + 1$ test [**2**].

Coverings are not the only tool in these investigations, since sometimes factorizations yield all the compositeness that is sought. Consider the situation with digit 1 but working in base $b = m^2$ with m odd. The minimal seed in all such cases is 1 because, for n appended 1s to the seed 1, with n even, the factorization

$$111\ldots11_b = \frac{b^{n+1} - 1}{b - 1} = \left(\frac{m^{n+1} - 1}{m - 1}\right)\left(\frac{m^{n+1} + 1}{m + 1}\right)$$

yields integer factors, and so the result is composite. When n is odd, the total number of 1s is even, so compositeness is clear. Similar factorization methods show that the minimal seed for digit 1 in base 4 is 5, for digit 3 in base 4 is 8, and for digit 8 in base 9 is 3.

3. A PANDIGITAL SEED.

It is not hard to find an integer that remains composite when any sequence of the form $ddd\ldots d$ is appended on the right, where d is any decimal digit. We leave it as an exercise to show that 6930 does the job; only the case $d = 1$ requires a prime cover, and the one used in §2 for 891—$(11, 37, 11, 3, 11, 13)$—works. Some prime searching shows that 6930 is the smallest such example (the most difficult candidate to eliminate was 6069; 1525 1s yielded a prime).

A more natural problem in our context is to consider only the digits 1, 3, 7, 9, and ask for an integer k that is a seed for each of these four digits (thus k is coprime to 3 and 7). We call such a positive integer k a *pandigital seed*.

For a prime p coprime to 10, we use the term *period* of p to mean the smallest positive integer r so that, for all n, $s_{n+r} \equiv s_n \pmod{p}$. The period of 3 is 3, while for other primes it is simply the order of 10 modulo p. If the period of a prime p is small, then p may divide a large proportion of the terms of the sequence s_n. In particular, if the period is r, then either every rth term of $\{s_n\}$ is divisible by p or no terms of the sequence are divisible by p.

Theorem. *A pandigital seed exists. An example is* 4942768284976776320.

Proof. A proof requires only checking that particular covers work, but we outline the method by which the large seed and corresponding prime covers were found. We find, for each digit, a prime cover so that the congruence conditions on k arising from the four covers do not contradict each other. This method of coherent prime covers was used in [**1, 4, 8**] to find infinitely many values k so that both $k2^n + 1$ and $k2^n - 1$ are composite for all n, and solve related problems. To find such covers, we first need to analyze the condition that a term in the sequence $\{s_n\}$ is divisible by a given prime p.

If we assume that $p \notin \{2, 3, 5\}$, then $s_n \equiv 0 \pmod{p}$ if and only if p divides

$$9k \cdot 10^n + d(10^n - 1),$$

which is equivalent to

$$k \equiv 9^{-1}d(10^{-n} - 1) \pmod{p}. \tag{1}$$

If $p = 3$, then we instead have the condition

$$s_n \equiv k + d\frac{10^n - 1}{9} \equiv 0 \pmod{3},$$

which, because $(10^n - 1)/9 \equiv n \pmod{3}$, reduces to $k \equiv 2dn \pmod{3}$. It is useful to observe that when n is even then $10^n \equiv 1 \pmod{11}$, so that in this case s_n is congruent modulo 11 to the seed itself. Therefore, the condition $k \equiv 0 \pmod{11}$ makes 11 a factor of s_n whenever n is even. Hence we may focus on forcing composites for odd values of n.

Since the period of $p = 37$ is 3, we consider this prime next. When the number of appended digits is $n = 6i + 3$, equation (1) gives

$$k \equiv 10^{-(6i+3)} - 1 = \left(10^{-6}\right)^i \cdot 10^{-3} - 1 \equiv 0 \pmod{37}.$$

Application of (1) to other values of n shows that 37 divides s_n for $n \equiv 0, 1, 2 \pmod{3}$ provided $k \equiv 0, 11d, 10d \pmod{37}$, respectively. If $k \equiv 0 \pmod{37}$, then 37 may be used as a prime divisor no matter which digit is appended. Therefore, we can assume $k \equiv 0 \pmod{37}$, and so we have that s_n is divisible by 11 when $n \equiv 0, 2, \text{or } 4 \pmod{6}$ and by 37 when $n \equiv 0 \text{ or } 3 \pmod{6}$. This leaves only the eight cases $n \equiv 1$ or 5 (mod 6) with digits 1, 3, 7, and 9 to be taken care of by other primes, as shown in Table 1.

To find divisors of s_n for $n \equiv 1$ or 5 (mod 6), we note that the primes 7 and 13 have period 6. Solving congruence (1) leads to the conditions listed in Table 2. These

Table 1. Divisors of s_n for digit d using primes 11 and 37 with a seed that satisfies $k \equiv 0 \pmod{11 \cdot 37}$.

	$n \pmod 6$					
digit	0	1	2	3	4	5
1	11	?	11	37	11	?
3	11	?	11	37	11	?
7	11	?	11	37	11	?
9	11	?	11	37	11	?

show that if $k \equiv 2 \pmod 7$, then two of the eight cases are divisible by 7: the digit 1 with $n \equiv 1 \pmod 6$ and digit 9 with $n \equiv 5 \pmod 6$ cases. Similarly, any of $k \equiv 1$, 3, or 9 (mod 13) provides divisibility for two of the cases. Each of these cases is then combined with a set of additional primes that contains 3, 101, 41, 271, 73, and 137, all of which have period 8 or less. Finally, a computer search found a list of primes that handles all cases.

Table 2. Conditions on k to guarantee that 7 or 13 divides the number obtained by appending a digit string to k.

	$n \equiv 1 \pmod 6$	$n \equiv 5 \pmod 6$	$n \equiv 1 \pmod 6$	$n \equiv 5 \pmod 6$
digit 1	$k \equiv 2 \pmod 7$	$k \equiv 1 \pmod 7$	$k \equiv 9 \pmod{13}$	$k \equiv 1 \pmod{13}$
digit 3	$k \equiv 6 \pmod 7$	$k \equiv 3 \pmod 7$	$k \equiv 1 \pmod{13}$	$k \equiv 3 \pmod{13}$
digit 7	$k \equiv 0 \pmod 7$	$k \equiv 0 \pmod 7$	$k \equiv 11 \pmod{13}$	$k \equiv 7 \pmod{13}$
digit 9	$k \equiv 4 \pmod 7$	$k \equiv 2 \pmod 7$	$k \equiv 3 \pmod{13}$	$k \equiv 9 \pmod{13}$

The smallest value of k found so far uses the primes 3, 7, 11, 13, 31, 37, 41, 73, 101, 137, 211, 241, and 271. The cover-lengths for the four digit-cases are 6, 6, 30, and 8, respectively. The prime covers for the four digits are as follows:

$$
\begin{aligned}
d = 1 &: (11, 3, 11, 37, 11, 13),\\
d = 3 &: (11, 13, 11, 37, 11, 7),\\
d = 7 &: (11, 3, 11, 37, 11, 271, 11, 3, 11, 37, 11, 41, 11, 3, 11, 37, 11, 31,\\
&\quad\ 11, 3, 11, 37, 11, 211, 11, 3, 11, 37, 11, 241),\\
d = 9 &: (11, 73, 11, 101, 11, 137, 11, 101).
\end{aligned}
$$

Tables 3 and 4 show the correspondence between the values of n and k for each digit. For example, when we are appending n 7s to k where $n \equiv 11 \pmod{30}$, we see that 41 divides s_n whenever $k \equiv 28 \pmod{41}$.

We apply the Chinese Remainder Theorem to all of the conditions on k in Tables 3 and 4 to find the pandigital seed $k = 4942768284976776320$. ■

Because k is not divisible by 3 or 7, we can add $k \equiv 1 \pmod{10}$ to the conditions used in the proof, which then gives us

Table 3. Residue classes for the seed k that guarantee the compositeness of s_n when 1 or 3 is appended.

digit 1

classes for n	classes for k
0 (mod 2)	0 (mod 11)
1 (mod 6)	2 (mod 3)
3 (mod 6)	0 (mod 37)
5 (mod 6)	1 (mod 13)

digit 3

classes for n	classes for k
0 (mod 2)	0 (mod 11)
1 (mod 6)	1 (mod 13)
3 (mod 6)	0 (mod 37)
5 (mod 6)	3 (mod 7)

Table 4. Residue classes for the seed k that guarantee the compositeness of s_n when 7 or 9 is appended.

digit 7

classes for n	classes for k
0 (mod 2)	0 (mod 11)
1 (mod 6)	2 (mod 3)
3 (mod 6)	0 (mod 37)
5 (mod 30)	0 (mod 271)
11 (mod 30)	28 (mod 41)
17 (mod 30)	20 (mod 31)
23 (mod 30)	106 (mod 211)
29 (mod 30)	7 (mod 241)

digit 9

classes for n	classes for k
0 (mod 2)	0 (mod 11)
1 (mod 8)	21 (mod 73)
3 (mod 4)	9 (mod 101)
5 (mod 8)	40 (mod 137)

$$1970728582053685108721 \pmod{1965785813768708332401 0},$$

a value of k that satisfies the theorem as stated in §1. This yields infinitely many such values.

4. OPEN PROBLEMS.

We conclude with some unsolved problems.

1. Find a number of 3s that can be appended to 817 to obtain a probable prime, thus completing the proof, modulo probable primes, that 4070 is the minimal seed for the digit 3.
2. Find a number of 9s that can be appended to 4420 to produce a prime.
3. Certify primality of 480 with 11330 7s appended and 851 with 28895 7s. Doing so would complete the digit-7 case.
4. Data for all bases up to 10 can be found at [**10**]. Similar problems exist for these bases.
5. Find a base-ten pandigital seed that is smaller than 4942768284976776320.
6. Investigate for various bases the situation where the appended digits come from a fixed sequence, as was done by Jones and White [**6**] for base ten.

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JON GRANTHAM has been a researcher at the Center for Computing Sciences since 1997. His previous publications addressed the existence of various types of pseudo primes. He lives in Maryland with his wife and their twin three-year-olds.
Institute for Defense Analyses, Center for Computing Sciences, 17100 Science Drive, Bowie, MD 20715
grantham@super.org

WITOLD JARNICKI worked at the Jagiellonian University (Kraków, Poland) until 2007. His research concentrated on algebraic geometry and complex analysis. Since 2007 he has been working as a software engineer at Google.
Google Kraków, Rynek Glowny 12, 31-042 Kraków, Poland
witoldjarnicki@google.com

JOHN RICKERT teaches at the Rose-Hulman Institute of Technology and writes about number theory and graph theory. In his spare time he researches baseball history and statistics.
Rose-Hulman Institute of Technology, 5500 Wabash Avenue, Terre Haute, IN 47803
rickert@rose-hulman.edu

STAN WAGON is recently retired from Macalester College. His books include *The Banach-Tarski Paradox*, *Mathematica in Action*, and *VisualDSolve*. Other interests include geometric snow sculpture, mountaineering, and mushroom hunting. He is one of the founding editors of Ultrarunning magazine, but now finds that covering long distances is much easier on skis than in running shoes.
Mathematics Department, Macalester College, St. Paul, MN 55105
wagon@macalester.edu